Ch-6 - System of Particles & Rotational Motion

* System of particles :-

→ The system which has 2 or more than 2 particles is called system of particles.

(i) Rigid body

→ The relative position b/w particles does not change] It is ideal concept but in some cases solid body can be taken as rigid body.

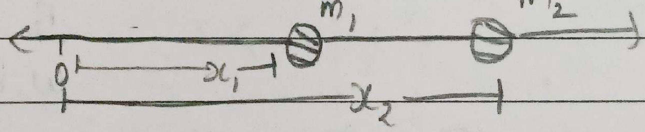
(ii) Non-rigid body

→ The relative position b/w particles change.

* Centre of mass :-

→ Mass-weighted mean is called centre of mass.

- For 1-D (2 Mass System)



→ Let 2 mass m_1 & m_2 whose position on x-axis is x_1 & x_2.

→ So C.M. of given system is defined as :-

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}$$

- For 1-D (3 particle System)

→ Let 3 particles of mass m_1, m_2 & m_3 with position x_1, x_2 & x_3. So C.M. of given

System is defined as :-

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1+m_2+m_3}$$

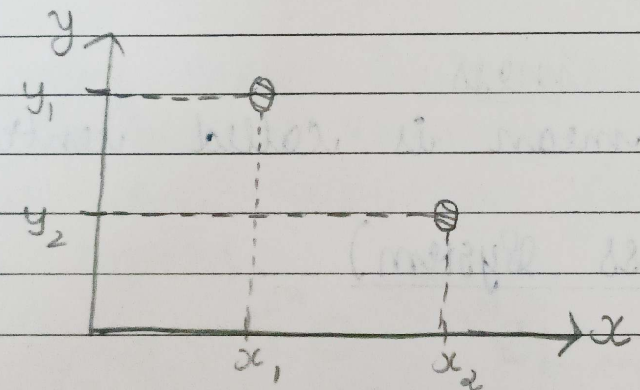
- For C.M. n-particle system we can write as

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1+m_2+\cdots+m_n}.$$

$\therefore$ $x_{cm} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i}$ where $\sum_{i=1}^{n} m_i = M$ = total mass of sytem

$\therefore$ $x_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_ix_i$

* For 2-D (2 mass system)



Let us consider 2 mass m_1 & m_2 with position

$\vec{r}_1 = (x_1, y_1)$

$\vec{r}_2 = (x_2, y_2)$.

w.r.t. origin.

→ Here x compo. of ~~c.m~~ c.m.

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}.$$

Similarly y compo. of c.m,

$$y_{cm} = \frac{m_1y_1 + m_2y_2}{m_1+m_2}$$

→ So, cm can be written as :

$$\vec{r}_{cm} = (x_{cm}, y_{cm})$$

Here ; $\vec{r}_{cm} = \left(\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} , \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \right)$

$$\vec{r}_{cm} = \frac{m_1 x_1 \hat{i} + m_2 x_2 \hat{i} + m_1 y_1 \hat{j} + m_2 y_2 \hat{j}}{m_1 + m_2}$$

$$= \frac{m_1 (x_1 , y_1) + m_2 (x_2 , y_2)}{m_1 + m_2}$$

$$\vec{r}_{cm} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2}$$

- For n- particle system we can write as:-

$$\vec{r}_{cm} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2} + \ldots + m_n \vec{r_n}}{m_1 + m_2 + \ldots + m_n}$$

$$\vec{r}_{cm} = \frac{\sum_{i=1}^{n} m_i \vec{r_i}}{\sum_{i=1}^{n} m_i}$$, But $\sum_{i=1}^{n} m_i = M$ = total mass of system

$$\vec{r}_{cm} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r_i}$$

→ x comp.

$$x_{cm} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i$$

→ y - compo.

$$y_{cm} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i$$

→ Z - compo.

$$z_{cm} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i$$

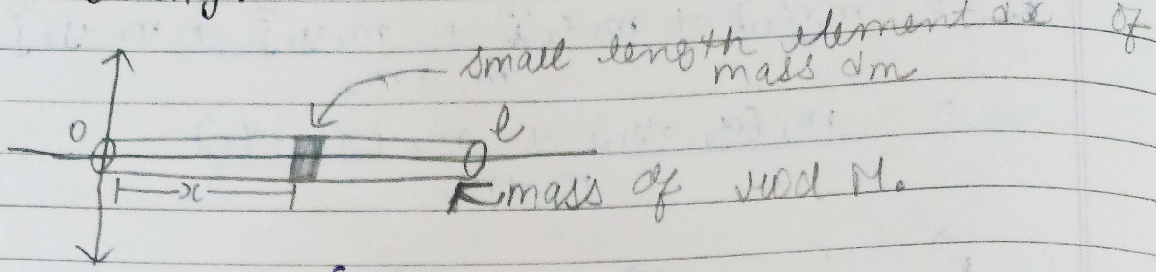
→ If body is continuous then,

$$x_{cm} = \frac{1}{M} \int x\,dm$$

$$y_{cm} = \frac{1}{M} \int y\,dm$$

$$z_{cm} = \frac{1}{M} \int z\,dm$$

* C.M. of l length rod with uniform mass density.



$$x_{cm} = \frac{1}{M}\int x\,dm$$

$l \Rightarrow M.$
$dx \rightarrow dm$ } $dm = \frac{M}{l} \times dx$

So, $x_{cm} = \frac{1}{M}\int_0^l x \times \frac{M}{l}\,dx$

$$= \frac{1}{l}\int_0^l x\,dx = \frac{1}{l}\left[\frac{x^2}{2}\right]_0^l = \frac{l^2}{2l} = \boxed{\frac{l}{2}}$$

* Extra (i) Linear mass density $(\lambda) = \frac{m}{l}$
(ii) Surface mass density $(\sigma) = \frac{m}{A}$
(iii) Volume mass density $(\rho) = \frac{m}{V}$

* Motion of C.M.

→ Let n-particles system with mass $m_1, m_2 \dots m_n$ & their position vector $\vec{r_1}, \vec{r_2} \dots \vec{r_n}$.
So, C.M. of given system.

$$\vec{r}_{cm} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \dots + m_n\vec{r_n}}{m_1 + m_2 + \dots + m_n}$$

Where, $m_1 + m_2 + \dots + m_n = M$ = total mass of system

$$\vec{r}_{cm} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \dots + m_n\vec{r_n}}{M}$$

$M\vec{r}_{cm} = m_1\vec{r}_1 + \cdots + m_n\vec{r}_n$

Take derivation w.r.t time

$\frac{d}{dt} M\vec{r}_{cm} = \frac{d}{dt} m_1\vec{r}_1 + \cdots + \frac{d}{dt} m_n\vec{r}_n$

Let mass of each particle remain unchange with time.

$M\frac{d\vec{r}_{cm}}{dt} = m_1\frac{d\vec{r}_1}{dt} + \cdots + m_n\frac{d\vec{r}_n}{dt}$

here $\frac{d\vec{r}_i}{dt} = \vec{v}_i$ = velocity of i^{th} particle

$\boxed{M\vec{v}_{cm} = m_1\vec{v}_1 + \cdots + m_n\vec{v}_n}$

Take derivation w.r.t. time.

$M\frac{d\vec{v}_{cm}}{dt} = \frac{m_1 dv_1}{dt} + \cdots + \frac{m_n dv_n}{dt}$

here $\frac{d\vec{v}_i}{dt} = \vec{a}_i$

= acceleration of i^{th} particle

$M\vec{a}_{cm} = m_1\vec{a}_1 + \cdots + m_n\vec{a}_n$

here $m\vec{a} = \vec{F}$ ($\because$ from newton's 2^{nd} law of motion)

$M\vec{a}_{cm} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n = \sum_{i=1}^{n} \vec{F}_i$

here $\vec{F}_i$ is vector addition of all force which are internal & external.

So, $M\vec{a}_{cm} = \vec{F}_{ext} + \vec{F}_{int}$

$(\because \sum_{i=1}^{n} \vec{F}_i = \vec{F}_{ext} + \vec{F}_{int})$

From newton's 3^{rd} law of motion,

$\vec{F}_{int} = 0$

So,

$\boxed{\underbrace{M}_{\text{total mass of system}}\overbrace{\vec{a}_{cm}}^{\text{acceleration of CM}} = \underbrace{\vec{F}_{ext}}_{\text{total ext. force acting on system.}}}$ → it is newton's 2^{nd} law of motion for system of particles.

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→ From this we can say all force act on system, but its acceleration is shown by acceleration of c.m. So, we can consider mass of system concentrated at c.m.

* Momentum of system of particles :-

→ c.m. of n-particles system

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_n\vec{r}_n}{m_1 + m_2 + \dots + m_n}$$

$$M\vec{r}_{cm} = m_1\vec{r}_1 + \dots + m_n\vec{r}_n .$$

$(m_1 + m_2 + \dots + m_n = M =$ total mass of system)

→ Take derivation w.r.t. time.,

$$\frac{M\,d\vec{r}_{cm}}{dt} = \frac{m_1\,d\vec{r}_1}{dt} + \dots + \frac{m_n\,d\vec{r}_n}{dt}$$

here $\frac{d\vec{r}_i}{dt} = \vec{v}_i$ = velocity of i^{th} particle.

$$M\vec{v}_{cm} = m_1\vec{v}_1 + \dots + m_n\vec{v}_n$$

here $m_i\vec{v}_i = \vec{P}_i$ = momentum of i^{th} particle.

$$M\vec{v}_{cm} = \vec{P}_1 + \dots + \vec{P}_n = \vec{P}_{total}$$

↳ Total momentum of system.

Take derivation w.r.t. time.

$$\frac{M\,d\vec{v}_{cm}}{dt} = \frac{d\vec{P}_1}{dt} + \dots + \frac{d\vec{P}_n}{dt} = \frac{d\vec{P}_{total}}{dt}.$$

here $\frac{d\vec{v}_{cm}}{dt} = \vec{a}_{cm}$

$$M\vec{a}_{cm} = \frac{d\vec{P}_{total}}{dt} \quad \text{but } M\vec{a}_{cm} = \vec{F}_{ext}$$

$$\frac{d\vec{P}_{total}}{dt} = M\vec{a}_{cm} = \vec{F}_{ext}$$

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* Conservation law of momentum.

→ For system of particles,

$$\frac{d\vec{P}_{total}}{dt} = \vec{F}_{ext}$$

if $\vec{F}_{ext} = 0$.

$$\frac{d\vec{P}_{total}}{dt} = 0.$$

So, $\vec{P}_{total}$ = constant.

→ If net force act on any system is zero then total linear momentum always remains constant. ← cons. law of momentum

here $\vec{P}_{total} = \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n$

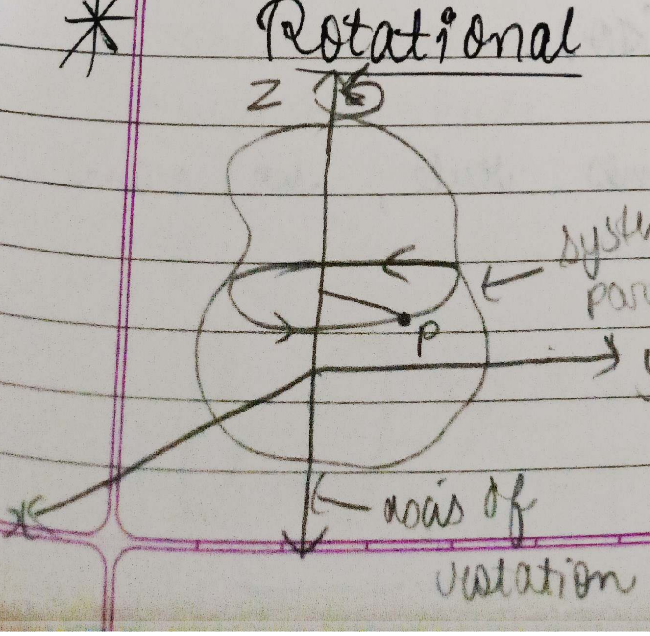
If $\vec{F}_{ext} = 0$

$$\frac{d\vec{P}_{total}}{dt} = 0 \Rightarrow \frac{d(\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n)}{dt} = 0$$

$$= \frac{d\vec{p}_1}{dt} + \cdots + \frac{d\vec{p}_n}{dt} = 0.$$

→ This means momentum of individual particle can change but total change is always zero if external force acting on system is zero. (This change happen by only internal forces).

* Rotational Motion :-



Let a body rotating about say z axis known as axis of rotation.

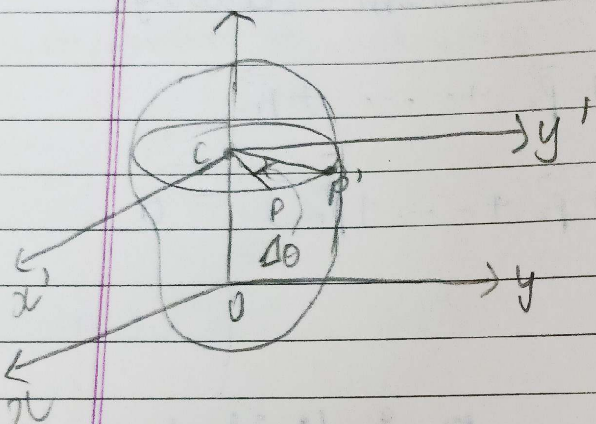
→ Due to rotation of body, each particles performs circular motion.

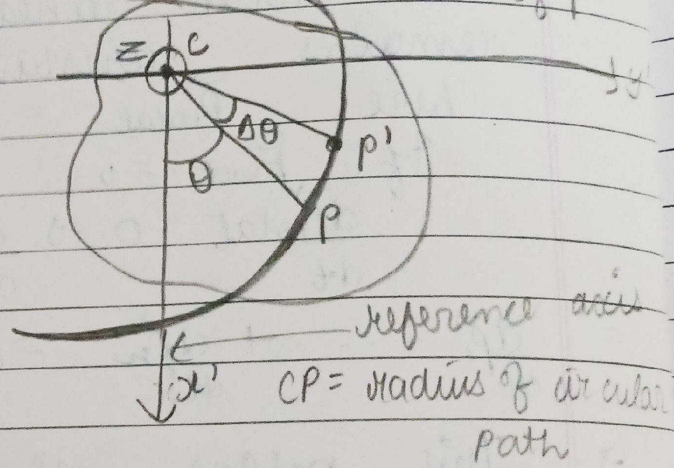
→ Here, circular plane formed by circular path of particles is always perpendicular to axis of rotation.

→ Centre of all circular path of particles always lie on axis of rotation.

→ All circular planes are parallel to each other.

* Angular position & displacement :-





→ Angle made by radius with any reference axis is called angular position (θ).
Unit = radian.

→ Change in angular position is known as angular displacement (Δθ).

→ It is vector quantity.

→ Using right hand thumb rule, we can identify it.

→ Unit = rad.

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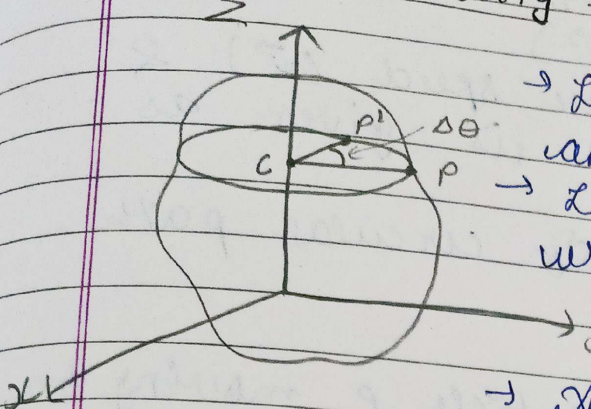
* Angular velocity :-



→ Let us consider a body rotating about axis say z-axis.
→ Let a particle of this body, whose position at 't' time is 'p'.

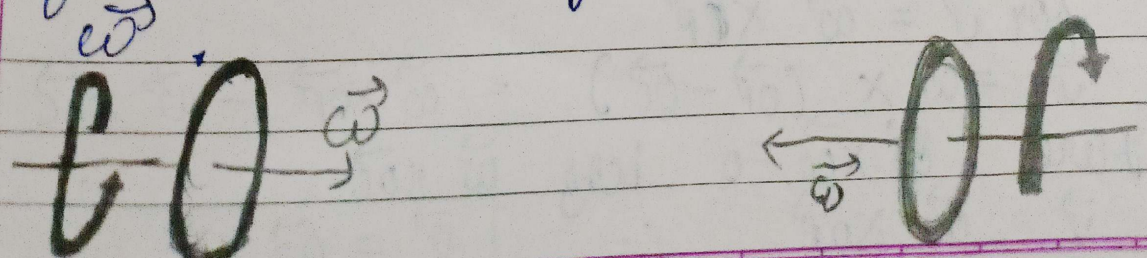
→ After some time Δt its position at $t + \Delta t$ time its position be p'.
→ Here angle b/w CP & CP' is $\Delta\theta$ known as angular displacement.
→ So, change in angular position w.r.t. time is known as angular speed (ω).

$$\langle\omega\rangle = \frac{\Delta\theta}{\Delta t}$$

for $\Delta t \to 0$, $\langle\omega\rangle \to \omega$

$$\therefore \omega = \frac{d\theta}{dt}$$

→ So, angular velocity can be given as, $\vec{\omega} = \frac{d\vec{\theta}}{dt}$
→ Here magnitude of angular velocity & angular speed is equal.
→ Here, $\vec{\omega}$ is along axis of rotation if axis of rotation is fixed.
→ We can identify it using right hand thumb rule. For that curl finger of right hand along rotation. Direction of thumb along axis show direction of $\vec{\omega}$



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* Relation b/w angular velocity ($\vec{\omega}$) & linear velocity ($\vec{v}$)

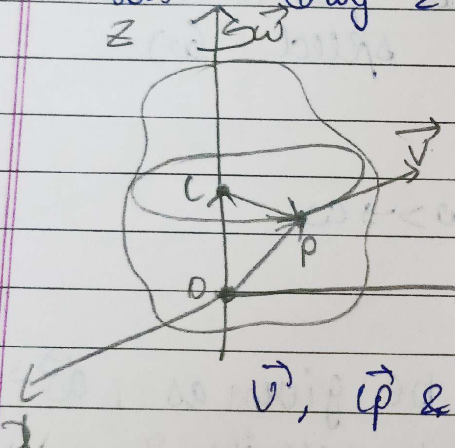
→ Relation b/w linear speed ($\vec{v}$) & Angular speed (ω) is given as :

$$v = r_\perp \omega$$

where $r_\perp$ = radius of circular path.

⇒ Let us consider a particle P moving on circular path of radius ~~CP~~ CP.

→ Here body is rigid body rotating with angular speed ω, about fixed axis say z axis.



So, linear speed of given particle P is $v = (CP)\omega$

Here $\vec{CP} \perp \vec{\omega}$

$v = (CP)\omega \sin 90$

here vector relation b/w $\vec{v}$, $\vec{cp}$ & $\vec{\omega}$ can be

$$\vec{v} = \vec{CP} \times \vec{\omega} \quad \text{or} \quad \boxed{\vec{v} = \vec{\omega} \times \vec{CP}}$$

Here, $\vec{\omega} \times \vec{CP}$ shows correct direction of $\vec{v}$ so $\vec{v} = \vec{\omega} \times \vec{CP}$ is correct relation.

→ Let position vector of P w.r.t. origin O is $\vec{OP} = \vec{r}$.

∴ Here $\vec{OP} = \vec{OC} + \vec{CP}$

$\vec{CP} = \vec{OP} - \vec{OC}$

So, for $\vec{v} = \vec{\omega} \times \vec{CP}$.

$\vec{v} = \vec{\omega} \times (\vec{OP} - \vec{OC}) = \vec{\omega} \times \vec{OP} - \vec{\omega} \times \vec{OC}$

Here $\vec{\omega} \times \vec{OC} = 0$ bcoz $\vec{\omega} \parallel \vec{OC}$.

So, $\vec{v} = \vec{\omega} \times \vec{OP}$ ∴ $\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$

* Angular momentum

→ Let consider a particle with mass m, moving with velocity $\vec{v}$.

→ If position vector of a particle is $\vec{r}$ w.r.t. origin 'O', then angular momentum of a particle is given as:

$$\vec{l} = \vec{r} \times \vec{p}$$, where $\vec{p}$ = linear momentum of a particle.

$= m\vec{v}$

- **Magnitude :-**

→ Magnitude of angular momentum

$l = rp \sin\theta$, where θ is angle b/w $\vec{r}$ & $\vec{p}$.

- **Direction :-**

→ Here direction of $\vec{l}$ is perpendicular to plane formed by $\vec{r}$ & $\vec{p}$ which is identified using right hand thumb rule.

here magnitude of $\vec{l}$ is $l = rp\sin\theta$

where

$r\sin\theta$ = perpendicular comp. of $\vec{r}$ to $\vec{p}$ $= r_\perp$

$p\sin\theta$ = perpendicular compo. of $\vec{p}$ to $\vec{r}$ $= p_\perp$

So, $l = rp_\perp$ or $l = r_\perp p$

* Relation b/w angular momentum & torque:-

→ Let a particle of mass m, moving with velocity $\vec{v}$, under action of force $\vec{F}$.

→ Here angular momentum of a particle w.r.t. origin 'O' is given as.

$$\vec{l} = \vec{r} \times \vec{p}$$

here, $\vec{r}$ = position vector of a particle w.r.t. O.

→ Take derivation w.r.t. time,

$$\frac{d\vec{l}}{dt} = \frac{d}{dr}(\vec{r} \times \vec{p})$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

→ Here $\frac{d\vec{r}}{dt} = \vec{v}$, $\vec{p} = m\vec{v}$
& $\frac{d\vec{p}}{dt} = \vec{F}$

So, $$\frac{d\vec{l}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}$$

$$\frac{d\vec{l}}{dt} = 0 + \vec{r} \times \vec{F}$$

here $\vec{r} \times \vec{F} = \vec{\tau}$

So, $\boxed{\frac{d\vec{l}}{dt} = \vec{\tau}}$ → This eqn look like as $\vec{F} = \frac{d\vec{p}}{dt}$.

So, $\vec{\tau} = \frac{d\vec{l}}{dt}$ is Newton's 2nd law of motion for rotational motion of particle.

* Relation b/w torque & angular momentum of system of particles :-

→ Let us consider a system of n-particles with angular momentum $\vec{l_1}$, $\vec{l_2}$ ..., $\vec{l_n}$

So, total angular momentum of a system

$$\vec{L} = \vec{l_1} + \vec{l_2} + \dots + \vec{l_n}$$

So,

$$\vec{L} = \Sigma \vec{l_i}$$

where, $\vec{l_i} = \vec{r_i} \times \vec{p_i}$

So, $\vec{L} = \Sigma \vec{r_i} \times \vec{p_i}$.

Take derivation w.r.t. time,

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\Sigma \vec{r_i} \times \vec{p_i})$$

$$\frac{d\vec{L}}{dt} = \Sigma \left[\frac{d}{dt}(\vec{r_i} \times \vec{p_i})\right]$$

So, $\frac{d\vec{L}}{dt} = \Sigma \left[\frac{d\vec{r_i}}{dt} \times \vec{p_i} + \vec{r_i} \times \frac{d\vec{p_i}}{dt}\right]$

$\frac{d\vec{r_i}}{dt} = \vec{v_i}$ & $\frac{d\vec{p_i}}{dt} = \vec{F_i}$ = force on i^{th} particle

$$\frac{d\vec{L}}{dt} = \Sigma \left[\underbrace{(\vec{v_i} \times \vec{p_i})}_{0 \,\because\, (\vec{v_i} \parallel \vec{p_i})} + (\vec{r_i} \times \vec{F_i})\right]$$

So,

$$\frac{d\vec{L}}{dt} = \Sigma (\vec{r_i} \times \vec{F_i}).$$

Here, $\vec{F_i}$ is vector addition of internal force & external force.

So, $\vec{F_i} = (\vec{F_i})_{int} + (\vec{F_i})_{ext}$.

So, $\frac{d\vec{L}}{dt} = \Sigma \left(\vec{r_i} \times \{(\vec{F_i})_{int} + (\vec{F_i})_{ext}\}\right)$

$$\frac{d\vec{L}}{dt} = \Sigma \left(\vec{r_i} \times (\vec{F_i})_{int} + \vec{r_i} \times (\vec{F_i})_{ext}\right)$$

Here $\Sigma \vec{r_i} \times (\vec{F_i})_{int} = \vec{\tau}_{int}$.

$\Sigma \vec{r_i} \times (\vec{F_i})_{ext} = \vec{\tau}_{ext}$.

So, $\frac{d\vec{L}}{dt} = \vec{\tau}_{int} + \vec{\tau}_{ext}$.

→ Here, according to 3^{rd} law of motion $\vec{\tau}_{int} = 0$

$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}}$ → this is equal to $\vec{F}_{ext} = \frac{d\vec{P}_{total}}{dt}$

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* Conservation law of angular momentum :-

→ We know,

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$$

Let $\vec{\tau}_{ext} = 0$

$$\frac{d\vec{L}}{dt} = 0 \qquad \text{so, } \vec{L} = \text{constant.}$$

→ If external torque acting on system of particle is zero, then total angular momentum of system always remains constant.

→ Here, $\vec{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k} = \text{const.}$

so,
$$L_x = \alpha = \text{const.} = \Sigma (l_i)_x$$
$$L_y = \beta = \text{const.} = \Sigma (l_i)_y$$
$$L_z = \gamma = \text{const.} = \Sigma (l_i)_z$$

* Equilibrium of Rigid body :-

→ If net force & net torque act on rigid body is zero then we can say our rigid body is in equilibrium.

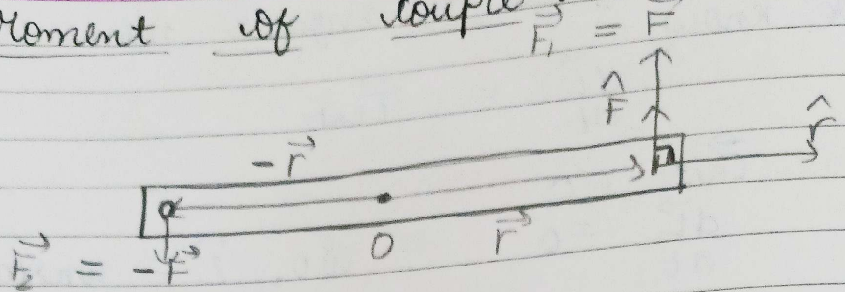
→ That mean if $\Sigma\vec{F} = 0$ — cond-1
& $\Sigma\vec{\tau} = 0$ — cond-2

→ If any one condition is fulfilled then that equilibrium is partial equilibrium.

(i) Translational equilibrium } $\Sigma\vec{F} = 0$ → $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_z = 0$

(ii) Rotational equilibrium } $\Sigma\vec{\tau} = 0$ → $\Sigma\tau_x = 0$, $\Sigma\tau_y = 0$, $\Sigma\tau_z = 0$

* Moment of couple :-



→ If equal & opposite forces which are not collinear then torque acting on system is known as moment of couple.

→ Here, net torque on system.

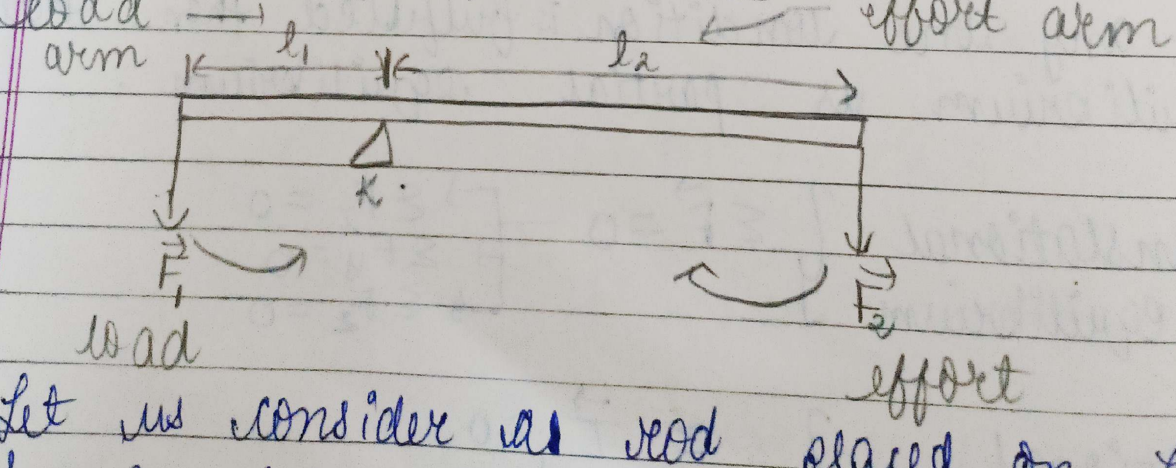
$$\vec{\tau} = \vec{\tau_1} + \vec{\tau_2}$$
$$= \vec{r} \times \vec{F} + (-\vec{r}) \times (-\vec{F})$$
$$= 2\vec{r} \times \vec{F}$$

→ Magnitude of torque.

$$\tau = 2rF \sin\theta$$
$$\tau = (2r)(F) \sin 90$$

$$\boxed{\tau = \text{magnitude of force} \times \text{perpendicular distance b/w \& force}}$$

* Principle of moments :-



→ Let us consider a rod placed on fulcrum & 2 force F_1 & F_2 act on this rod as per diagram.

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→ Let rod is in equilibrium,
So, for rotational equilibrium w.r.t. k

$$\Sigma \tau = 0$$

$$+F_1 \times l_1 - F_2 l_2 = 0$$

$$F_1 \times l_1 = F_2 l_2$$

$$F_1 = \frac{F_2 \times l_2}{l_1}$$ } principle of moment or liver

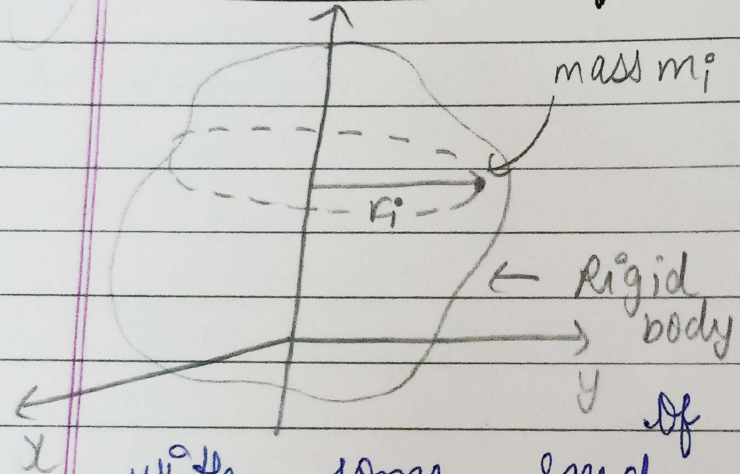
So, $F_1 > F_2$.

So, using less effort we can pull heavy load.

$$\frac{F_1}{F_2} = \frac{l_2}{l_1} = MA$$

mechanical advantage

* Moment of Inertia :-



→ Let us consider a system of particle with mass m_1, m_2, m_3 ....

→ Due to rotational motion of system all particles moving with some speed let it be v_1, v_2, v_3, ...

→ So, total K.E. of a system,

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \dots$$

$$K = \frac{1}{2} \Sigma m_i v_i^2$$

, If angular speed of i^{th} particle ω_i then,

$$v_i = r_i \omega_i$$

So,

So, $v_i = r_i \omega_i$

$\therefore$ $K = \frac{1}{2} \Sigma m_i r_i^2 \omega_i^2$

So, $K = \frac{1}{2} (\Sigma m_i r_i^2) \omega^2$

Compare it with $K = \frac{1}{2} m v^2$

$F = \frac{m\,dv}{dt} = \frac{dp}{dt}$

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So, we can say $\Sigma m_i r_i^2$ has equivalent property to mass, which is called moment of inertia.

$$\boxed{I = \Sigma m_i r_i^2}$$

where r_i is + distance of particle from axis of rotation.

So, $K = \frac{1}{2} I\omega^2$

'I' depends on mass distribution about axis of rotation.

- Extra

$$I = \Sigma m_i r_i^2$$

Let all particles have equal mass,

$$I = m \Sigma r_i^2$$

→ multiply / divide with n = no. of particles.

$$I = \frac{nm \cdot \Sigma r_i^2}{n}, \quad nm = M = \text{total mass}$$

$$I = M\left(\frac{\Sigma r_i^2}{n}\right)$$

where $\frac{\Sigma r_i^2}{n} = k^2$ = Radius of gyration

$$I = Mk^2$$

$$k^2 = \frac{\Sigma r_i^2}{n} = \frac{r_1^2 + r_2^2 + \dots + r_n^2}{n}$$

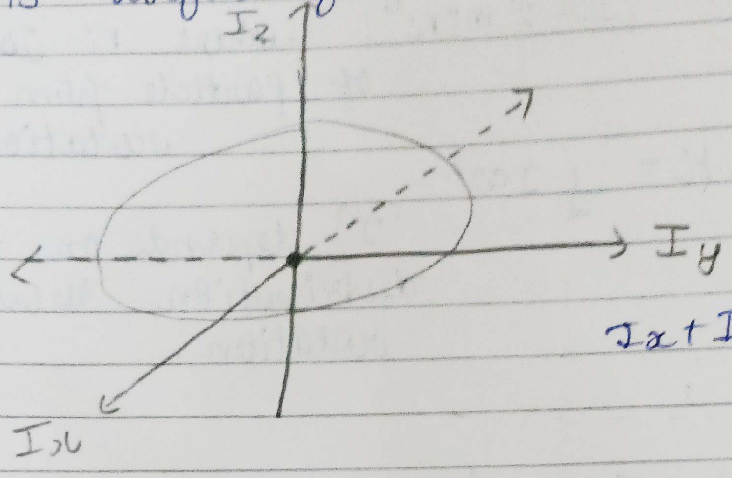
So, $k = \sqrt{\frac{r_1^2 + r_2^2 + \dots + r_n^2}{n}}$

$= r_{rms}$ square, mean, root

k = rms - value of perpendicular dist. from axis.

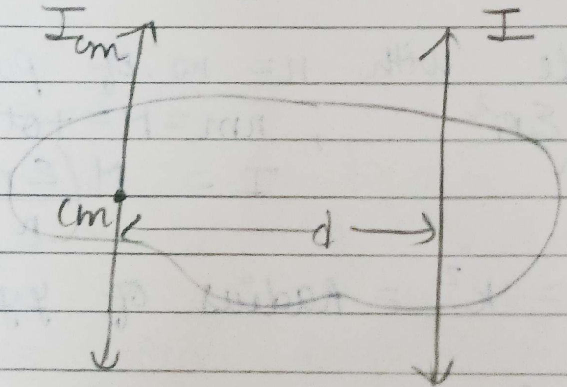
* Theorem of perpendicular axis.

→ It is useful for planer body.



$$I_x + I_y = I_z$$

* Theorem of parallel axis



$$I = I_{cm} + Md^2$$

* Kinematics of rotational motion about fixed axis.

Kinematic eq^n	eq^n for Rotation	
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$	ω_0 = angular speed at $t=0$
$x = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	ω = angular speed at time t.
$v^2 - v_0^2 = 2ax$	$\omega^2 - \omega_0^2 = 2\alpha\theta$	θ = angular displacem for time 't'
$\left(\frac{v+v_0}{2}\right)t = x$	$\frac{\omega+\omega_0}{2} \times t = \theta$.	α = const. angular acceleration.

* Deduce $\omega = \omega_0 + \alpha t$ using calculus method.

We know $\alpha = \frac{d\omega}{dt}$

$d\omega = \alpha dt$

take integration on both sides

$$\int_0^{\omega} d\omega = \int_0^{t} \alpha dt$$

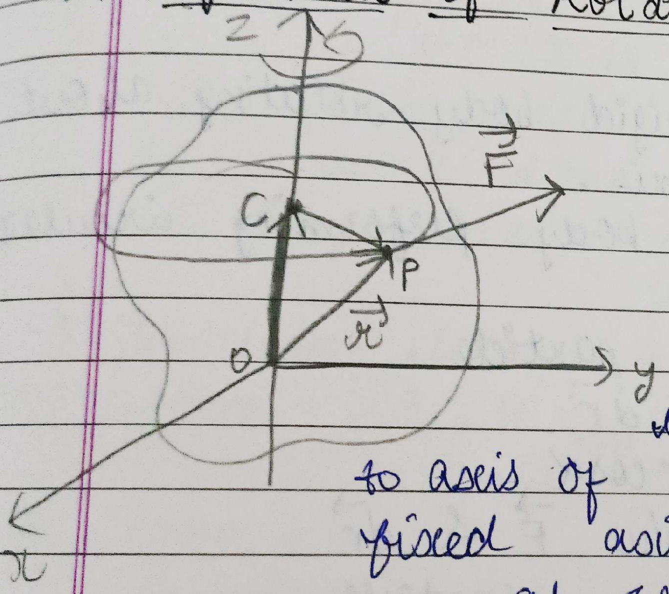
$$[\omega]_{\omega_0}^{\omega} = \alpha [t]_0^t$$

$$\omega - \omega_0 = \alpha(t-0)$$

$$\omega - \omega_0 = \alpha t.$$

$$\boxed{\omega = \omega_0 + \alpha t}$$

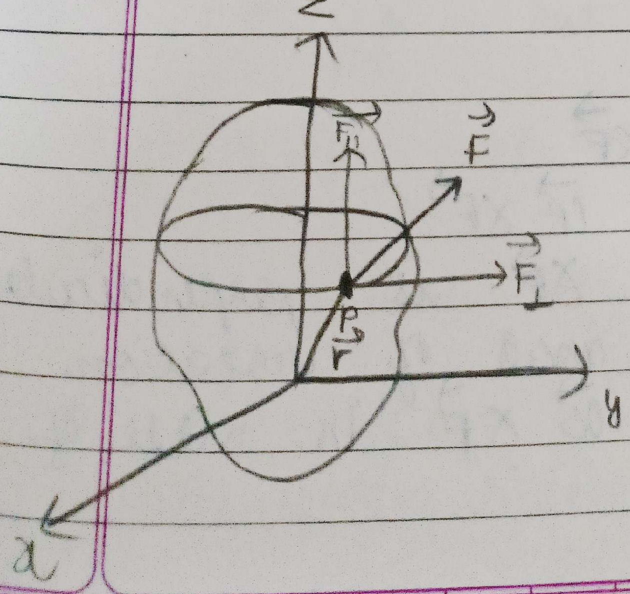
* Dynamics of Rotational motion about fixed axis.



Here torque of a particle

$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$= (\vec{OC} + \vec{CP}) \times \vec{F}$$
$$= (\vec{OC} \times \vec{F}) + (\vec{CP} \times \vec{F})$$
$$= (\vec{OC} \times \vec{F}) \perp \vec{OC} \text{ (along axis of rotation)}$$

compo. of torque $\vec{OC} \times \vec{F}$ is $\perp$ to axis of rotation but in case of fixed axis $\vec{OC} \times \vec{F}$ can be taken as zero.



Here torque $\vec{\tau} = \vec{r} \times \vec{F}$

$$= \vec{r} \times (\vec{F}_{\perp} + \vec{F}_{\parallel})$$

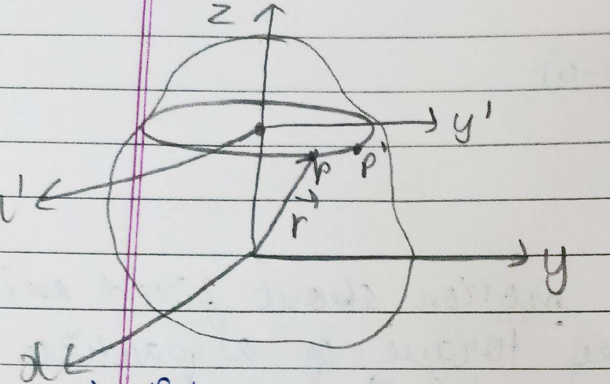
perpendicular compo. of force to the axis of rotation

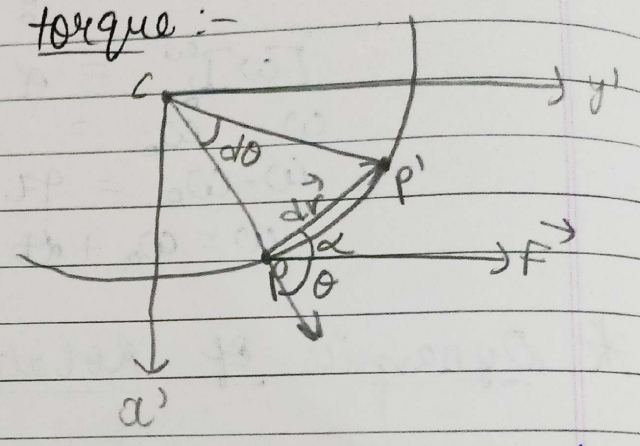
parallel compo. to axis of rotation.

$$\vec{\tau} = (\vec{r} \times \vec{F_\perp}) + (\vec{r} \times \vec{F_\parallel})$$

→ Here compo. of torque $\vec{r} \times \vec{F_\parallel}$ is $\perp$ to $\vec{F_\parallel}$
So, $\vec{r} \times \vec{F_\parallel}$ is $\perp$ to axis of rotation
But in case of rotation about fixed axis $\vec{r} \times \vec{F_\parallel}$ can be taken as zero.

* Work done by torque :-





→ Let us consider a rigid body rotating about fixed axis say z axis.
→ A particle of this body performing circular motion force $\vec{F}$
→ work done on this particle.

$$dW = \vec{F} \cdot d\vec{r}$$

$$dW = F\,dr\cos\alpha$$

where α is angle b/w $\vec{F}$ & $d\vec{r}$

→ Now, torque on given particle.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

but $\vec{r} = \vec{OP} = \vec{OC} + \vec{CP}$.

So, $\vec{\tau} = (\vec{OC} + \vec{CP}) \times \vec{F}$

So, $\vec{\tau} = \vec{OC} \times \vec{F} + \vec{CP} \times \vec{F}$

Here torque due to $\vec{OC} \times \vec{F}$ is perpendicular to $\vec{\alpha}$ that means $\perp$ to axis of rotation
So, we can neglect $\vec{OC} \times \vec{F}$ in case of fixed axis.

So, $\vec{\tau} = \vec{CP} \times \vec{F}$

$\vec{\tau} = (CP) F \sin\theta$ (θ is angle b/w $\vec{CP}$ & $\vec{F}$).

For $\Delta PCP'$,

$d\theta = \frac{dr}{CP} \Rightarrow dr = d\theta \times CP$

So, work can be written as.

$dw = F dr \cos\alpha$

$= F \times d\theta \times CP \cos\alpha$

But $\theta + \alpha = 90^\circ \Rightarrow \alpha = 90 - \theta$

$dW = (F)(CP) d\theta \cos(90-\theta)$

$dW = F \times CP \, d\theta \sin\theta$.

But $\tau = (CP) F \sin\theta$

So,

$dW = \tau d\theta$.

For given rigid body total work is addition of work done on each particle.

So, total work.

$W = \int dw = \int \tau d\theta$

→ It look like as work done by variable force $W = \int F dx$

* Power :-

Power = rate of work done

$P = \frac{dw}{dt}$ but $dw = \tau d\theta$.

So, $P = \frac{\tau d\theta}{dt}$

Here $\frac{d\theta}{dt} = \omega$

$\Rightarrow P = \tau\omega$.

It look like as $P = FV$.

→ Rotational K.E.

$K_R = \frac{1}{2} I\omega^2$

Take derivation w.r.t. time

$\frac{dK_R}{dt} = \frac{1}{2} I \frac{d\omega^2}{dt}$

$\frac{dK_R}{dt} = \frac{1}{2} I \times 2\omega \frac{d\omega}{dt}$

where, $\frac{dK_R}{dt} = P = \tau\omega$

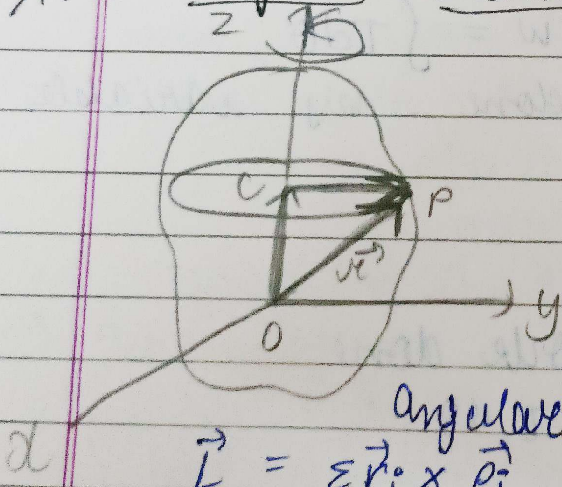
$\frac{d\omega}{dt} = \alpha$

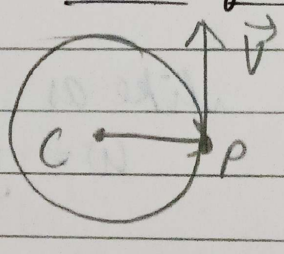
So, $P = \tau\omega = I\omega \times \alpha$

So, $\boxed{\tau = I\alpha}$

It look like as $\boxed{F = ma}$

* Angular momentum about fixed axis :-





Let us consider a system of n- particles, in that total angular momentum of system

$\vec{L} = \Sigma \vec{r}_i \times \vec{p}_i$

For any particle 'p'.

$\vec{l} = \vec{r} \times \vec{p}$

Here $\vec{r} = \vec{OP} = \vec{OC} + \vec{CP}$

So $\vec{l} = (\vec{OC} + \vec{CP}) \times \vec{p}$

$\vec{l} = \vec{OC} \times \vec{p} + \vec{CP} \times \vec{p}$

(For fixed axis 'p' is always ⊥ to axis of rotation)

Here $\vec{OC} \times \vec{p}$ is perpendicular to axis of rotation.
So, $\vec{OC} \times \vec{p}$ = $l_\perp$ of component of $\vec{l}$, which is perpendicular to axis of rotation.
Here $\vec{CP} \times \vec{p}$ is parallel to axis of rotation
So, $\vec{CP} \times \vec{p}$ = l_z of component of $\vec{l}$ parallel to axis of rotation means z axis.

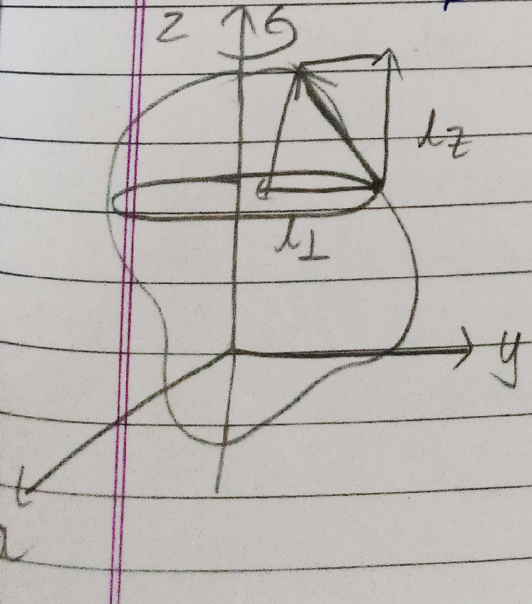
So, total angular momentum of system,

$$\vec{L} = \Sigma l_\perp + \Sigma l_z$$

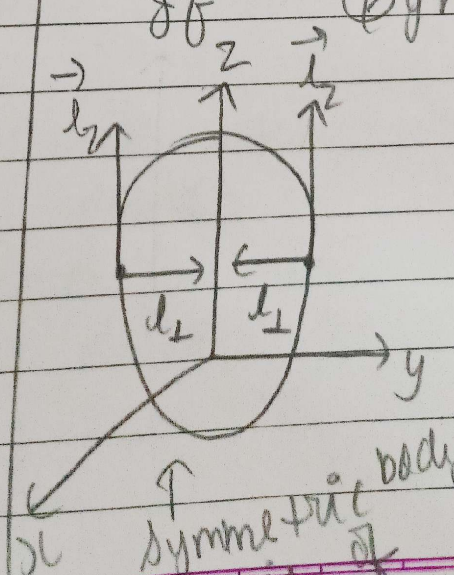
$$\vec{L} = \vec{L}_\perp + \vec{L}_z$$

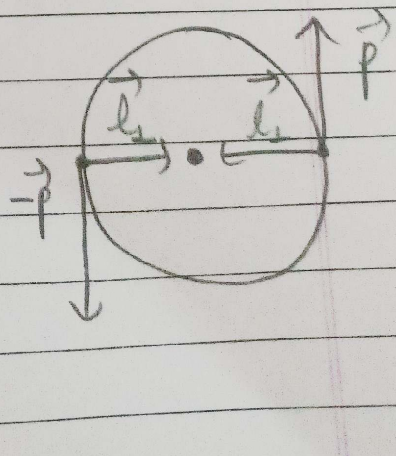
Here $\vec{L} = \vec{L}_\perp + \vec{L}_z$
that means it is not necessary that angular momentum always along axis of rotation.
Now, for symmetric body about axis of rotation $\vec{L}_\perp$ be zero because every particle has symmetric particle whose compo. of angular momentum which are mutually opposite.
That means, for symmetric body,

$$\vec{L} = \vec{L}_z = \Sigma \vec{l}_z = \Sigma \vec{CP} \times \vec{p}$$



If Symmetric





We know $\vec{L} = \vec{L_z} = \Sigma \vec{CP} \times \vec{p}$

$= \Sigma [(CP)\, p \sin 90]\, \hat{k}$

here $CP = r_\perp$ = radius of circular path.

$p = mv = m \times r_\perp \omega$

So, $\vec{L} = \Sigma (r_\perp m r_\perp \omega)\, \hat{k}$

For rigid body,

$\vec{L} = (\Sigma m r_\perp^2)\, \omega \hat{k}$

Here,

$\Sigma m r_\perp^2 = I$ = moment of Inertia.

$\vec{L} = \vec{L_z} = I\omega \hat{k}$

For symmetric body,

$\boxed{\vec{L} = I\vec{\omega}}$ → It look like as : $\vec{p} = m\vec{v}$

* Conservation law of angular momentum

§ Initial angular momentum = final angular momentum.

$I_1 \omega_1 = I_2 \omega_2$