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Ch - 7 - Gravitation

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1) => Ptolemy } Geocentric model
complex

2) => Plato } heliocentric model } Sun is centre
supported by Galileo

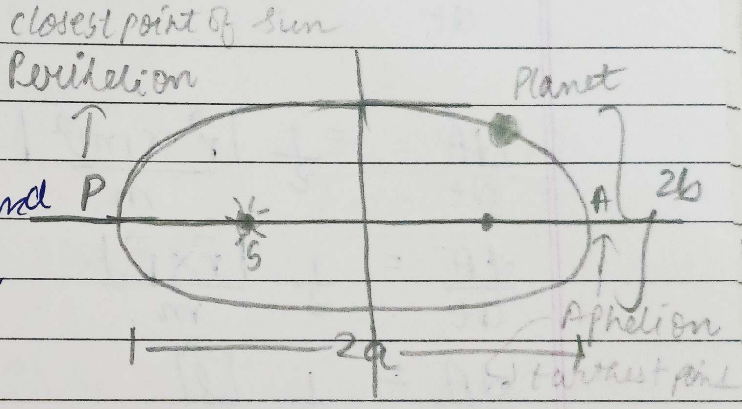
→ orbit of planets → circular

3) => Tyco Brahe } to observe motion of extraterrestrial body } using obs. of his
↳ Johannes Kepler
→ 1) L. of orbits
→ 2) L. o. areas
→ 3) L. o. periods

★ Kepler's Law :-

1) Law of orbits :-

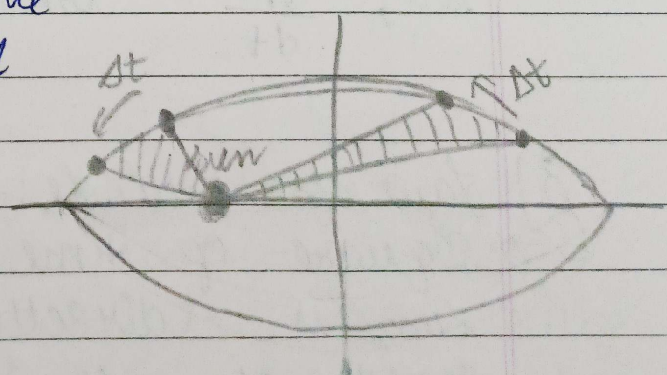
→ All planet revolve around sun in elliptical orbit, where sun is situated at one of the focii of elliptical orbit.



AP = 2a = Major Axis

2) Law of areas :-

→ Swept area by line joining the planet and sun is equal in equal time interval

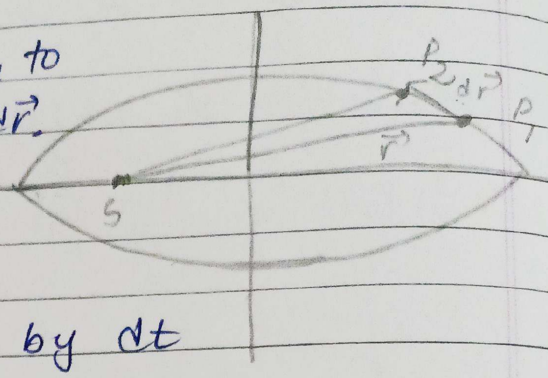


→ Speed of planet is max. at perihelion point

→ Speed of planet is min. at aphelion point.

- Prove that areal velocity of planets is constant.

→ Let planet move from P_1 to P_2 so. disp. occured is $d\vec{r}$.



area of $\Delta P_1 P_2 S$ is.

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}|$$

divide by dt

$$\frac{dA}{dt} = \frac{1}{2}\left|\vec{r} \times \frac{d\vec{r}}{dt}\right|$$

$$\frac{dA}{dt} = \frac{1}{2}|\vec{r} \times \vec{v}|$$

multiply & divide with mass of planet

$$\frac{dA}{dt} = \frac{1}{2}\frac{|\vec{r} \times m\vec{v}|}{m}$$

$$\frac{dA}{dt} = \frac{1}{2}\frac{|\vec{r} \times \vec{p}|}{m} \qquad \text{Here } \vec{r} \times \vec{p} = \vec{L}$$

$$\frac{dA}{dt} = \frac{1}{2}\frac{|\vec{L}|}{m}$$

here torque is zero on planet $|\vec{L}|$ is constant.

So, $\frac{dA}{dt}$ = constant

3) <u>Law of Periods</u>:- $T^2 \propto a^3$

→ Square of time period of revolution around sun is directly proportional to cube of length of semi-major axis.

→ Let time period of planets be 'T'. & length of semi major axis be 'a'.

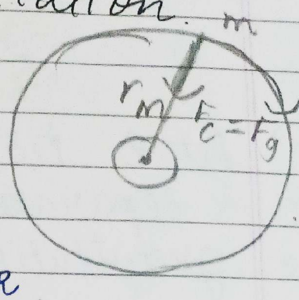
$T^2 \propto a^3$

* Universal Law of Gravitation

$\Rightarrow a_c = \frac{v^2}{r_m}$ $\qquad T = 27.3$ days

$v = \frac{2\pi r_m}{T}$ $\qquad r_m = 3.85 \times 10^8$ m

$\qquad a_c =$



$\therefore a_c = g$

$\Rightarrow a_c = \frac{4\pi^2 r_m^2}{r_m T^2}$ $\qquad \frac{4\pi^2 \times 3.85 \times 10^8}{(27.3 \times 24 \times 3600)^2}$

$= \frac{4\pi^2 r_m}{T^2}$ $\qquad = 0.0027$ m/s^2

$$a \propto r^n$$

$$\frac{9.8}{0.0027} = \frac{(6.4 \times 10^6)^n}{(3.85 \times 10^8)^n}$$

$$\therefore 3629.62 = \left(\frac{1.66}{100}\right)^{n}$$

$a \propto r^{-2}$

$a \propto \frac{1}{r^2}$

$$3629.62 = \frac{1}{(0.0166)^{+|n|}}$$

$n = -|n|$

$\boxed{n = -2}$

$F = ma$ $\qquad F_{12} = m_1 a_1$

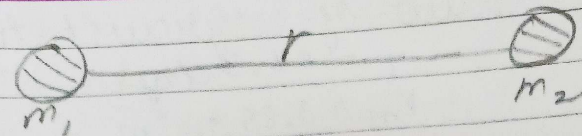
$F \propto a$ $\qquad F_{21} = m_2 a_2$

$F \propto \frac{1}{r^2}$ $\qquad \therefore F \propto m_1 m_2$

$$\therefore F \propto \frac{m_1 m_2}{r^2}$$

* Newton's law of Gravitation:-

→ All bodies in universe exert attractive type force on each others. The force acting between 2 bodies is directly proportional to multiplication of their mass & inversely proportional to square of distance b/w them.



→ Let 2 bodies with mass m_1 & m_2 be separated by distance r., then gravitational force b/w them is
$$F \propto \frac{m_1 m_2}{r^2}$$

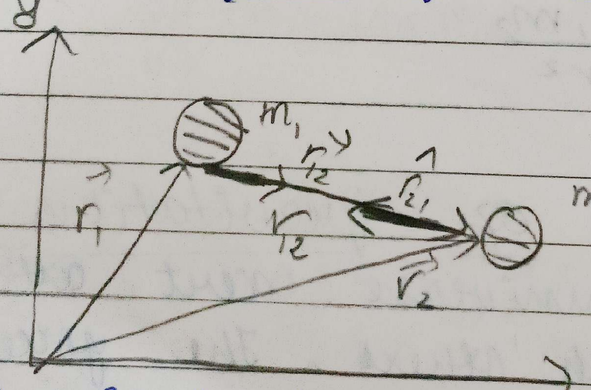
⇒ $$F = \frac{G m_1 m_2}{r^2}$$

where G = ~~grava~~ universal gravitational constant.
$$= 6.67 \times 10^{-11} \ \frac{Nm^2}{Kg^2}.$$

$[G] = M^{-1}L^3T^{-2}$

→ Gravitational force is long range force.
→ It does not depends on medium.

✱ Vector form of law of gravitation :-



→ Let 2 bodies of mass m_1 & m_2 separated by distance r.
→ Then force on m_1 due to m_2
$$F_{12} = \frac{G m_1 m_2}{r_{12}^2}$$

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Here r_{12} = dist. between m_1 & m_2

⇒ In vector form

$$\vec{F}_{12} = \frac{G m_1 m_2}{r_{12}^2} \hat{r}_{12}$$

Here $\hat{r}_{12}$ is unit vector from m_1 to m_2

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}} \leftarrow \text{vector joining } m_1 \text{ to } m_2$$

⇒ Using triangle method of vector addition

$$\vec{r}_1 + \vec{r}_{12} = \vec{r}_2$$

So, $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$

⇒ $|\vec{r}_{12}| = r_{12} = |\vec{r}_2| - |\vec{r}_1|$

Similarly force on m_2 due to m_1.

$$\vec{F}_{21} = \frac{G m_1 m_2}{r_{21}^2} \hat{F}_{21}$$

where $\hat{r}_{21} = \frac{\vec{r}_{21}}{r_{21}} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$.

⇒ Here $\vec{F}_{21} = \frac{G m_1 m_2}{r_{21}^2} \hat{r}_{21}$

Here $\hat{F}_{21} = -\left(\frac{G m_1 m_2}{r_{12}^2} \hat{r}_{12}\right)$

$\boxed{\vec{F}_{21} = -\vec{F}_{12}}$ → That means universal law of gravitation is relevant to newton's 3rd law of motion.

Principle of Superposition :-

→ If there are multiple gravitational force act on any mass then net gravitational force is vector sum of individual forces.

Let 3 masses m₁, m₂ & m₃ . So, net force on m_3 is

$$\vec{F_3} = \vec{F_{31}} + \vec{F_{32}}$$

$$= \frac{G m_1 m_3}{r_{31}^2} \hat{r}_{31} + \frac{G m_2 m_3}{r_{32}^2} \hat{r}_{32}$$

* Gravitational Intensity (field)

Unit → N/kg

So, force on M due to m.

$$F = \frac{GMm}{r^2}$$

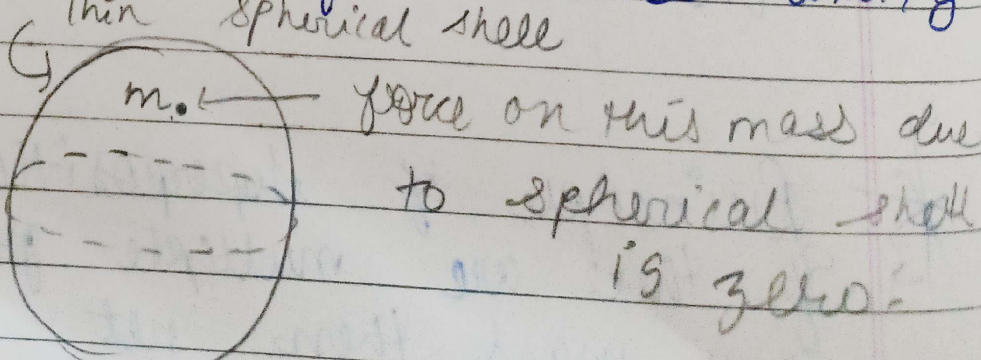
⇒ G.I. = $\frac{\text{gravitational force}}{\text{mass}}$

$$I = \frac{GM\cancel{m}}{r^2 \times \cancel{m}} \quad \therefore \quad I = \frac{GM}{r^2}$$

$$I = \frac{F}{m} \quad \therefore \quad \boxed{F = Im}$$

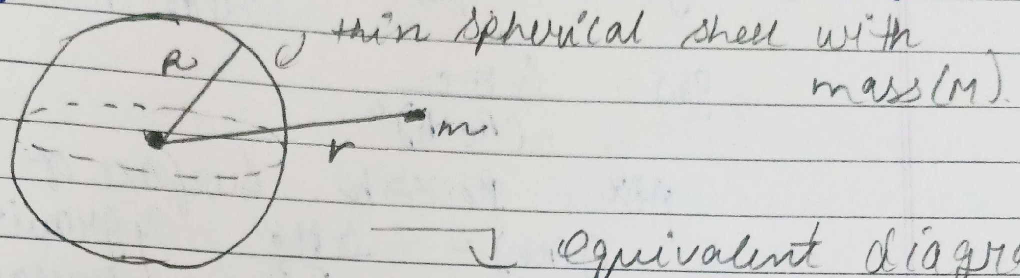
* I^st Theorem

⇒ Force on point mass reside inside thin spherical shell with uniform mass density is zero.

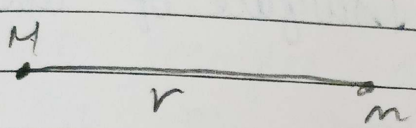


* 2nd Theorem :-

→ To find force ~~of~~ on point mass which reside outside the thin spherical shell with uniform mass density. We can consider whole mass of shell concentrated at its centre.



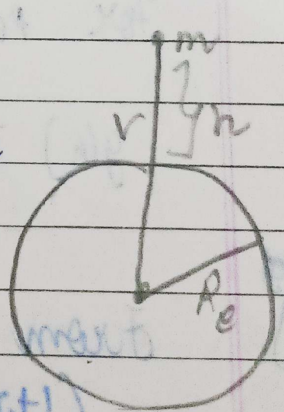
equivalent diagram



So, force on m due to M due to M. is

$$F = \frac{GMm}{r^2}$$

* Gravitational acceleration varies with altitude (height)



Let m mass at outside earth.

→ Here we consider earth as thin concentric spherical shell

→ And for outside we can consider all mass of shell concentrated at its centre.

→ That means whole mass of earth can be considered as concentrated at centre of earth.

→ So, gravitational force due to earth on 'm' mass $F = \frac{G m_e m}{r^2}$, m_e = mass of earth

r = distance of 'm' mass from centre of earth.

→ So, gravitational acceleration

$$g(r) = \frac{F}{m}$$

$$g(r) = \frac{G m_e m}{r^2 \times m}$$

$$g(r) = \frac{G m_e}{r^2}$$ Here $r = R_e + h$.

$$g(h) = \frac{G m_e}{(R_e + h)^2}$$

for $R_e \gg h$. Surface of earth.

$$g(R_e) = \frac{G M_e}{R_e^2}$$ } gravitational acc^n at surface of earth.

⇒ For $R_e \gg h$

$$g(h) = \frac{G m_e}{R_e^2 (1 + h/R_e)^2}.$$

$$g(h) = g(R_e) \left(1 + \frac{h}{R_e}\right)^{-2}.$$

For $R_e \gg h$ ⇒ $\left(1 + \frac{h}{R_e}\right)^{-2} = 1 - \frac{2h}{R_e}$.

$$\boxed{g(h) = g(R_e)\left(1 - \frac{2h}{R_e}\right)}$$

⇒ From Binomial Theorem.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots$$

$(x < 1)$

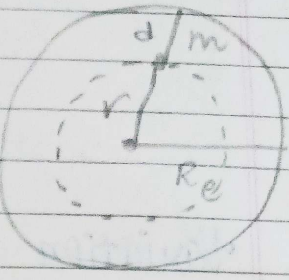
So for

$$\left(1 + \frac{h}{R_e}\right)^{-2} = 1 + (-2)\left(\frac{h}{R_e}\right) + \ldots$$ } Terms which contain 2 or more than 2 power neglected.

$$= 1 - \frac{2h}{R_e}$$

* Gravitational acceleration varies with depth.

→ To find gravitation acc^n at 'd' depth from surface of earth.



→ Assume point mass 'm' at 'r' distance from centre of earth such that $r = R_e - d$

→ Here gravitational force due to 'd' thickness shell is zero. So, we get gravitational force only due to 'r' radii sphere of earth.

→ So, we consider mass of 'r' radii sphere at centre of earth. So, force due to it on 'm' mass

$$F = \frac{GMm}{r^2} \quad (M = \text{mass of } r \text{ radii sphere}).$$

So, gravitational acc^n

$$g(r) = \frac{F}{m} = \frac{GM\cancel{m}}{r^2 \times \cancel{m}}$$

$$g(r) = \frac{GM}{r^2}$$

For $\frac{4}{3}\pi R_e^3 \rightarrow M_e$

$$\frac{4}{3}\pi r^3 \rightarrow M = \frac{\cancel{\frac{4}{3}\pi} r^3 \times M_e}{\cancel{\frac{4}{3}\pi} R_e^3} = \frac{M_e r^3}{R_e^3}$$

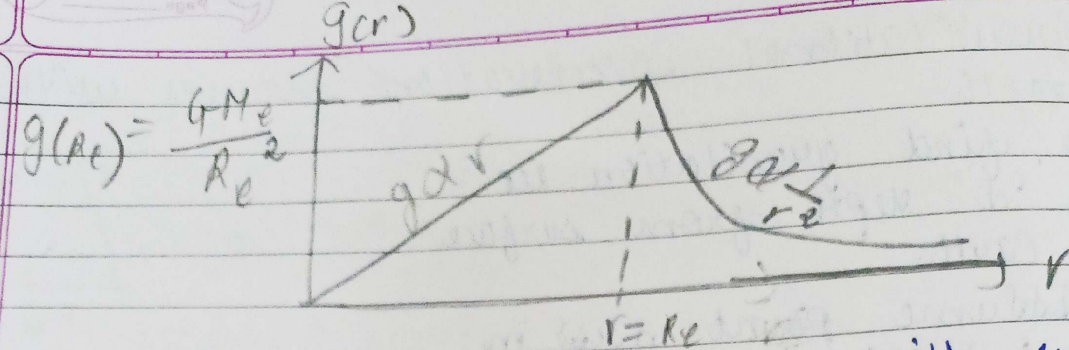
→ So, $g(r) = \frac{G M_e r^{\cancel{3}}}{R_e^3 \times \cancel{r^2}}$

$$g(r) = \frac{GM_e}{R_e^3} r$$

here $r = R_e - d$.

$$g(d) = \frac{GM_e}{R_e^2} \cdot \frac{R_e - d}{R_e}$$

$$\boxed{g(d) = g(R_e)\left(1 - \frac{d}{R_e}\right)}$$



Variation of gravitational acc^n with distance.

* Gravitational Potential Energy :-

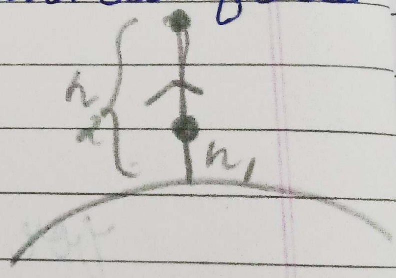
→ Near surface of earth gravitational acc^n $g = 9.8\ m/s^2$ is const.

→ So, gravitational force on body of mass m is $F_g = -mg$ (-ve sign bcoz it is downward).

→ Here, we move 'm' mass from h_1 to h_2 height from surface of earth.

→ So, work done by gravitational force
$$W_g = F_g d = (-mg)(h_2 - h_1)$$

→ So, P.E. $\Delta U = -W_g$



$$\Delta U = +mg(h_2 - h_1)$$
$$U(h_2) - U(h_1) = mgh_2 - mgh_1$$

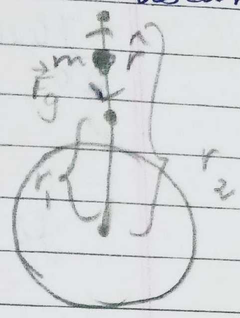
→ So, P.E. at 'h' height is
$$U(h) = mgh + W_0$$
where W_0 is constant.

but difference of P.E. is meaningful so, W_0 can be taken arbitarily zero.

So, $U(h) = mgh$

→ Let we move m mass without any acceleration from r_1 to r_2, where r_1 & r_2 is distance from centre of earth.

→ At any distance 'r' from centre of earth gravitational force on 'm' mass by earth



$$\vec{F_g} = \frac{-G m_1 m_2}{r^2}\hat{r}$$

Now, work done by gravity,

$$W_g = \int \vec{F_g} \cdot d\vec{r}$$

here, $d\vec{r} = dr\,\hat{r}$

→ So,

$$W_g = -\int \frac{G M_e m}{r^2}\hat{r} \cdot dr\,\hat{r}$$

$\hat{r} \cdot \hat{r} = 1$

So, $W_g = -G m_e m \int_{r_1}^{r_2} r^{-2}\,dr$

$$= -G m_e m \left[\frac{r^{-1}}{-1}\right]_{r_1}^{r_2}$$

$$W_g = G m_e m \left[\frac{1}{r}\right]_{r_1}^{r_2}$$

$$W_g = \frac{G m_e m}{r_2} - \frac{G m_e m}{r_1}$$

So, P.E is

$$\Delta U = -W_g$$

$$U(r_2) - U(r_1) = -\left[\frac{G m_e m}{r_2} - \frac{G m_e m}{r_1}\right]$$

$$U(r_2) - U(r_1) = \frac{-G m_e m}{r_2} - \left(-\frac{G m_e m}{r_1}\right) \quad \text{—— (1)}$$

So, $U(r) = \frac{-G m_e m}{r} + W_0$

Here, difference of P.E. is meaningful
So, W_0 can be taken arbitarily zero.

So, $U(r) = -\frac{G m_e m}{r}$

If we take $r = \infty$, $U(\infty) = 0$

Let $r_2 = r$ & $r_1 = \infty$

$$U(r) - U(\infty) = -\frac{G m_e m}{r} - \left(-\frac{G m_e m}{\infty}\right)$$

$$\boxed{U(r) = -\frac{G m_e m}{r}}$$

Gravitational P.E. at any point is (-ve) workdone of gravitational force to move 'm' mass from '∞' to 'r'.

- extra :-

⇒ Gravitational intensity = $\frac{\text{gravitational force}}{\text{mass}}$

$$I = \frac{F}{m} = \frac{GM\cancel{m}}{r^2 \times \cancel{m}}$$

$$\boxed{I = \frac{GM}{r^2}}$$

⇒ Gravitational potential = $\frac{\text{Potential energy}}{\text{mass}}$

$$= \frac{U}{m} = \frac{-GMm}{r \times m}$$

$$= \boxed{-\frac{GM}{r}}$$

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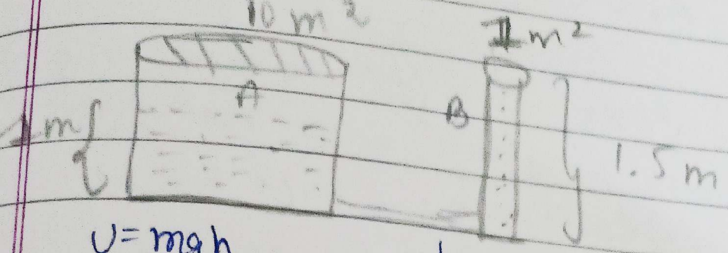
• extra:-

→ gravitational potential near surface of earth $= \frac{U}{m} = \frac{mgh}{m}$

$= \boxed{gh}$



$U = mgh$
$= V\rho gh$
$= 10 \times 1 \times 10^3 \times 10 \times 1$
$= 1 \times 10^5$ J.
$gh = 10 \times 1 = 10$

$U = mgh$
$= V\rho gh$
$= 1 \times 1.5 \times 10^3 \times 10 \times 1.5$
$= 2.25 \times 10^4$
$= 0.225 \times 10^5$ J.
$gh = 1.5 \times 10 = 15$

✱ escape speed & escape energy:-

→ escape energy } minimum energy required to free the object from gravitational influence.

so,
energy at surface of earth (E_{R_E}) = energy at infinite (E_∞)

$\Rightarrow \frac{-GM_Em}{R_E} + W_0 = 0$

so, $\boxed{W_0 = \frac{GM_Em}{R_E}}$

(escape energy)

This energy is always in form of K.E.

$\therefore \frac{1}{2}mv_e^2 = \frac{GM_Em}{R_E}$

$$V_e^2 = \frac{2GM_E}{R_E} \Rightarrow V_e = \sqrt{\frac{2GM_e}{R_e}}$$

$$V_e = \sqrt{\frac{2GM_e}{R_e^2} \times R_e}$$

$$V_e = \sqrt{2gR_E} \quad \left(\because \frac{GM_e}{R_e^2} = g\right)$$

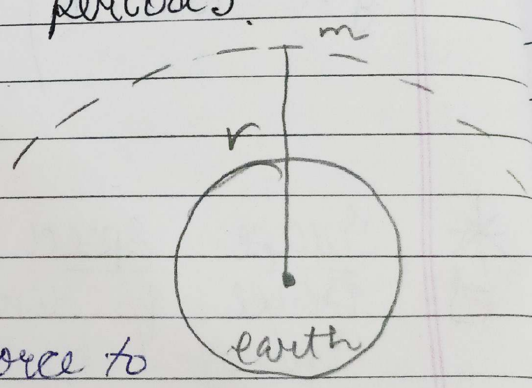
$\Rightarrow$ For earth $V_e = \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$

$= 11.2 \times 10^3$ m/s

$= 11.2$ km/s

* earth satellite (Time period)

$\rightarrow$ Let consider a satellite of mass 'm' moving around earth in orbit of radius r from centre of earth.



$\rightarrow$ Here required centripetal force to revolve satellite around earth is provided by gravitational force of earth

$$F_c = F_g$$

$$\Rightarrow \frac{mv^2}{r} = \frac{GM_e m}{r^2}$$

$$\boxed{V^2 = \frac{GM_e}{r}} \quad \text{--- (1)}$$

Now, Speed $= \frac{\text{distance}}{\text{time}}$

$$V = \frac{2\pi r}{T}$$

$$T = \frac{2\pi r}{V} \quad \text{(Squaring on both side}$$

$$T^2 = \frac{4\pi^2 r^2}{v^2}$$

$$T^2 = \frac{4\pi^2 r^2}{\frac{GM_e}{r}}$$ (from (1))

$$T^2 = \frac{4\pi^2}{GM_e} r^3$$

Here $T^2 \alpha r^3$ is form of Keplar's law of period.

Here $r = R_e + h$

where h = height of satellite from surface

$$T^2 = \frac{4\pi^2}{GM_e} (R_e + h)^3$$

→ Here time period of satellite does not depend on mass of satellite.

* Time of satellite which is near surface of earth ($h \approx 0$)

$$T^2 = \frac{4\pi^2 R_e^3}{GM_e} = \frac{4\pi^2 R_e^2 \times R_e}{GM_e}$$

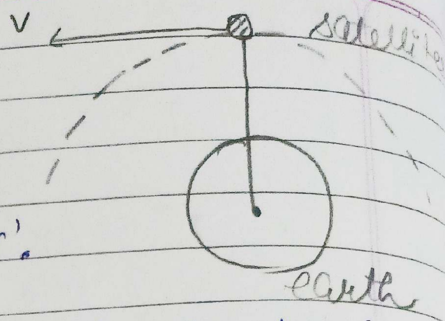
$$T^2 = \frac{4\pi^2 \times R_e}{g} = \frac{4\pi^2 \times 6.4 \times 10^6}{9.8}$$

$$T = 5.07 \times 10^3$$

$$T = 5070 s$$

$$T = 84.5 \text{ min}$$

* Energy of satellites



→ Let us consider a satellite of mass 'm' revolving around earth in orbit of radius 'r'.

→ Here satellite is revolving about earth in gravitational influence.

→ So, total energy of satellite,

$$E = K + U$$

potential energy of satellite $= U = \frac{-Gm_em}{r}$

K.E. of satellite $= K = \frac{1}{2}mv^2$

→ Here, required centripetal force to satellite revolving around earth is provided by gravitational force.

$$F_c = F_g$$

$$\frac{mv^2}{r} = \frac{GM_em}{r^2}$$

$$mv^2 = \frac{Gm_em}{r}$$

$$\frac{1}{2}mv^2 = \frac{Gm_em}{2r}$$

→ So, K.E. of satellite $= K = \frac{Gm_em}{2r}$

Total energy

$$E = K + U$$

$$= \frac{Gm_em}{2r} - \frac{Gm_em}{r}$$

$$E = -\frac{G m_e m}{2r}$$

→ If distance of satellite from surface of earth is h.

$r = R_e + h$

So, $E = \frac{-G m_e m}{2(R_e + h)}$

• Extra :-

$$\boxed{K = -E = -\frac{U}{2}}$$